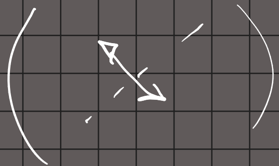
Math 261B Tues 10/6

Reminder $SO_N \subset SL_N$ preserves $\underset{\in k^N}{(x, y)} = x^T J y$

$$J = \begin{pmatrix} 0 & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}$$

$A \in SO_N \iff A^R A = I$ and $\det(A) = 1$

$A^R = J A^T J$



$T \subset B \subset SO_N$

$$T = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \qquad N = 2n$$

$$\begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1 & & & \\ & & & & t_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix} \qquad N = 2n+1$$

$B =$ upper Δ'ar

$X = X(T) = \mathbb{Z}^n$

Lie algebra $\mathfrak{so}_N$: $M^R + M = 0$
(char $k \neq 2$)

$$\begin{pmatrix} & x & 0 \\ & \ddots & -x \\ 0 & & \end{pmatrix} \qquad \begin{pmatrix} \cdot & x & y & z & \\ & \cdot & & & -z \\ & & \ddots & & -y \\ & & & \cdot & -x \\ & & & & \cdot \end{pmatrix}$$

SO_{2n+1}

$T \curvearrowright (-)$ $\quad x \longmapsto t_i/t_j \, x$

root $e_i - e_j \in \mathbb{Z}^n$

$\underline{t}^{e_i - e_j} = t_i/t_j$

$y \mapsto t_i y$ $\quad$ root e_i $\quad t_i = \underline{t}^{e_i}$

$z \longmapsto t_i t_j z$ $\quad$ root $e_i + e_j$ $\quad (i \neq j)$

Pos roots : $\quad e_i - e_j \quad (i<j)$

$R_+ \qquad e_i + e_j$

$\qquad e_i$

$R = R_+ \sqcup -R_+ = \{\pm e_i \pm e_j, \pm e_i\} \subseteq \mathbb{Z}^n$

Root SL_2's

$$\underset{SL_2}{\overset{\cap}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \longmapsto \begin{pmatrix} a & b & & & \\ c & d & & & \\ & & \ddots & & \\ & & & a & -b \\ & & & -c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right)^R = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$X^* = (\mathbb{Z}^n)^* \cong \mathbb{Z}^n \quad \varepsilon_i$

$\langle \varepsilon_i, e_j \rangle = \delta_{ij}$

$\varepsilon_i - \varepsilon_j = \text{coroot}$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \qquad \begin{pmatrix} t_i = t & & & \\ & t_j = t^{-1} & & \\ & & t_j^{-1} = t & \\ & & & t_i^{-1} = t^{-1} \end{pmatrix}$$

$t_i = t$

$t_j = t^{-1}$

$t_k = 1$ otherwise

$\begin{matrix} d & c \\ b & a \end{matrix}$

All $\underset{\alpha^\vee}{\pm\varepsilon_i \pm \varepsilon_j}$ are coroots $\longleftrightarrow$ $\underset{\alpha}{\pm e_i \pm e_j}$

$\langle \alpha^\vee, \alpha \rangle = 2$

$\langle \varepsilon_i - \varepsilon_j , e_i - e_j \rangle = 2$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

" " "

$SL_2 \overset{Ad}{\curvearrowright} sl_2 = \langle E, H, F \rangle$

$tr(XY) = tr(YX)$

$tr(EF) = 1 \qquad tr(H^2) = 2$

$2E, H, F \qquad \frac{1}{2} tr(XY) \qquad (H,H) = 1$

$(2E, F) = 1$

$SL_2 \longrightarrow PSL_2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & -ab & -b^2/2 \\ -2ac & ad+bc & bd \\ -2c^2 & 2cd & d^2 \end{pmatrix} \in SO_3$$

$$\begin{pmatrix} & & \vdots & & \\ & (i,i)\ \circ & \circ & \circ & \\ - - - & \circ & \circ & \circ & - - - \\ & \circ & \circ & \circ & \\ & & \vdots & & \end{pmatrix}$$

$SO_3 \subset SO_{2n+1}$

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & t^2 \ (t_i) & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & 0 & & & & t^{-2} \ (t_i^{-1}) & \\ & & & & & & 1 \end{pmatrix}$$

$t_i \to t^2$

$t_j \to 1$

$\underset{\alpha^\vee}{2\varepsilon_i} \longleftrightarrow \underset{\alpha}{e_i}$

$$\begin{array}{llll} & \alpha_1 & \alpha_{n-1} & \alpha_n \\ \text{Simple roots} & e_1 - e_2, \ldots, & e_{n-1} - e_n, & e_n \\ \text{coroots} & \varepsilon_1 - \varepsilon_2, \ldots & \varepsilon_{n-1} - \varepsilon_n, & 2\varepsilon_n \\ & \alpha_1^\vee & & \alpha_n^\vee \end{array}$$

$e_i \pm e_j \quad i<j$

e_i

$e_i - e_j$

$e_i - e_{i+1} + (e_{i+1} - e_{i+2}) + \cdots (e_{j-1} + e_j)$

$e_i - e_n + e_n = e_i$

$e_i + e_j = e_i - e_j + 2e_j$

$s_i = s_{\alpha_i}$ is $e_i \leftrightarrow e_{i+1}$ (generate S_n) $(i = 1, \ldots, n-1)$

$s_n = s_{\alpha_n}$ is $e_n \leftrightarrow -e_n$

$W = \{\text{signed permutations}\} = B_n$

$= \text{hyperoctahedral group}$

$= S_n \ltimes \{\pm 1\}^n$

$|W| = 2^n n!$

Cartan matrix $\langle \alpha_j^\vee, \alpha_i \rangle$ $\quad \langle \varepsilon_j - \varepsilon_{j+1}, e_i - e_{i+1} \rangle$

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & 0 & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}$$

$\langle \alpha_n^\vee, \alpha_{n-1} \rangle = \langle 2\varepsilon_n^\vee, e_{n-1} - e_n \rangle = -2$

$\langle \alpha_{n-1}^\vee, \alpha_n \rangle = \langle \varepsilon_{n-1} - \varepsilon_n, e_n \rangle = -1$

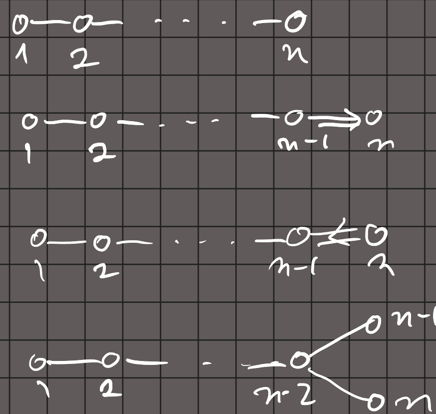
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \circ \quad \circ$ $\qquad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \circ\!-\!\circ$

$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad \circ \Rightarrow \circ$ $\qquad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \quad \circ \Rrightarrow \circ$

Dynkin diagram

B_n

$(SL_{n+1}, \cdots)$



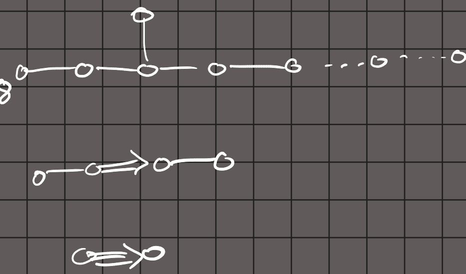
A_n

B_n — SO_{2n+1}

C_n — Sp_{2n}

D_n — SO_{2n}

classical groups



$E_{6,7,8}$

F_4

G_2

exceptional groups

SO_{2n+1} $\quad Q = X \Rightarrow SO_{2n+1}$ is <u>adjoint</u> $\quad X/Q = 1 \Rightarrow Z(SO_{2n+1}) = 1$

$Q^\vee \subseteq X^\vee \cong \mathbb{Z}^n$ $\quad SO_{2n+1} \overset{Ad}{\curvearrowright} so_{2n+1}$ is faithful

$\quad \shortparallel$

$\{\beta \in \mathbb{Z}^n \mid \Sigma \beta_i \text{ is even}\}$

$SO_{2n+1} \curvearrowright K^{2n+1}$

$\mathrm{Spec}(K \cdot X/Q) = Z(G)$

$\mathcal{O}(Z(G)) = K \cdot (X/Q)$

Another lattice embedding of the

"same" root system : $X^* = Q^\vee$ $\quad X = (Q^\vee)^* \supset Q \cong \mathbb{Z}^n$

$\mathbb{Z}^n \subset \mathbb{Q}^n$ $\quad (\frac{1}{2}, \dots, \frac{1}{2}) \in (Q^\vee)^* \subset \mathbb{Q}^n$

$(Q^\vee)^* = \mathbb{Z}^n \amalg (\mathbb{Z}^n + (\frac{1}{2}, \dots, \frac{1}{2})) \subseteq \mathbb{Q}^n$

Could take $X \supset Q$ $\quad \longrightarrow$ What's the group? $\quad X/Q \cong \mathbb{Z}/2\mathbb{Z}$

$X^* = Q^\vee$

$1 \to \{\pm 1\} \to Spin_{2n+1} \to SO_{2n+1} \to 1$

$Spin_{2n+1} \to SO_{2n+1}$ is double cover

$Spin_{2n+1}(\mathbb{C}) \to SO_{2n+1}(\mathbb{C})$ (universal b/c $X^* = Q^\vee$)

is the simply connected covering covering group.

$2n+1 = 3$

$n = 1$

Ex. $SL_2 \to SO_3$ $\quad SL_2 = Spin_3$ $\quad SO_3 \cong PSL_2$

$SL_2 \dashrightarrow PSL_2 \xrightarrow{\cong} SO_3$

$X = \mathbb{Z}$ $\quad X^* = \mathbb{Z}$

$Spin_3(\mathbb{R})$ $\quad SO_{2n+1}(\mathbb{R})$

$\downarrow$ $\quad \Uparrow$

$SO_3(\mathbb{R})$ $\quad$ with $(x, y) = x^t y$

$\alpha \quad \alpha^\vee$

$\langle \alpha^\vee, \alpha \rangle = 2$ $\quad (2)$

$SO_N(\mathbb{R})$ = rotations of unit ball in $\mathbb{R}^N$